J. Fluid Mech. (1966), vol. 26, part 1, pp. 17–35
Printed in Great Britain

On the steady Krook kinetic equation: Part 1

By DONALD G. ANDERSON

Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts

(Received 15 October 1965)

In this paper systems of non-linear integral equations are formulated which are equivalent to the Krook kinetic equation for steady problems in two and three space-dimensions. The boundary conditions used are discussed and some of the properties of the equations which are significant in their numerical solution pointed out. This work will then serve as a basis for the consideration of a sequence of particular problems to be presented in subsequent papers.

1. Introduction

It is the purpose of the present paper to formulate, and to discuss in general terms, systems of non-linear integral equations equivalent to the Krook kinetic equation for steady problems in two and three space-dimensions. This work will then serve as a basis for the consideration of a sequence of particular examples of such problems in subsequent papers of this series.

In the next section, we shall outline some of the background and motivation for such a study. In §§3 and 4, the integral-equation formulation of the steady Krook kinetic equation for problems in two and three space-dimensions will be derived. This is followed by a discussion of the boundary conditions employed. Finally, we shall describe a particular example and point out some of the properties of the equations which are significant in their numerical solution. This will lead us to the specification of a sequence of problems, whose solution, it is hoped, will elucidate some of the structure of the general equations.

2. Preliminary remarks

Perhaps the most active area of research in theoretical gas dynamics and kinetic theory at the present time is the interface between macroscopic, continuum theories and the more fundamental (in principle) microscopic, statistical theories. On the one hand, the Navier–Stokes equations of conventional gas dynamics can be modified and extended to reflect the insights of kinetic theory; on the other hand, the Maxwell–Boltzmann equation for the molecular velocity distribution function can be approximated by macroscopic continuum equations derived therefrom. (These two approaches are not mutually exclusive.)

Seemingly, the Navier–Stokes equations are valid much further into the rarefied gas régime than their approximate kinetic theoretical derivation would suggest, but are known to be inadequate to describe narrow transition regions in the flow such as boundary, initial and shock layers. The relationship of these layers to the 'continuum' flow in which they are embedded is somewhat analogous to the régimes of validity of the Navier–Stokes and Euler equations in conventional fluid mechanics. One of the most fruitful areas for research in this field would seem to be the various attempts which have been, and are being, made to connect a kinetic theoretical description of the transition region to the adjacent continuum. The alternative approach of deriving continuum-equation approximations from the kinetic equation is much less general in the sense that a good deal of insight into the particular class of problems at hand is required, but excellent results can be obtained with relatively simple models (Anderson & Macomber 1965; & Macomber 1965). Virtually all sensible approximation procedures embody the conservation of mass, momentum, and energy; these provide such powerful constraints that such gross effects as shock thickness, drag and heat-transfer coefficients can be rather insensitive to the details of the approximation procedure employed. This is an admirable situation for those concerned with the theoretical and empirical investigation of such quantities, but not from the point of view of those concerned with assessing the efficacy of, and differentiating between, approximation procedures. Consequently, a class of model problems for which both exact and approximate solutions can be obtained offers a very attractive avenue for investigating approximation procedures of all kinds. Such a class of model problems is available through the use of the Krook kinetic equation as an approximation to the Maxwell–Boltzmann equation. There is reason to believe that the essential physical phenomena are reasonably well approximated by the Krook equation through the whole range of Knudsen numbers.

The steady Krook kinetic equation for a simple gas of identical molecules with no internal degrees of freedom, in the absence of external fields, can be written (Krook 1955, 1959)

$$\lambda \mathbf{v} . \partial f / \partial \mathbf{x} = \nu(\mathbf{x})(F - f). \tag{1}$$

The distribution function $f(\mathbf{v};\mathbf{x})$ is proportional to the probability density of molecular velocities $\mathbf{v}$, as a function of position $\mathbf{x}$. The left side of the equation represents the convective rate of change of f; this is balanced by the rate of change of f due to molecular interactions, which is represented by the term on the right. $F(\mathbf{v};\, \mathbf{x})$ is the Maxwellian distribution given by

$$F(\mathbf{v};\, \mathbf{x}) = n(2\pi T)^{-\frac{3}{2}} \exp -\{(\mathbf{v}-\mathbf{q})^2/2T\}. \tag{2}$$

n, $\mathbf{q}$ and T are the local number density, flow velocity and kinetic temperature respectively, defined by the equations

$$n = \int d\mathbf{v} f(\mathbf{v};\, \mathbf{x}), \tag{3}$$

$$n\mathbf{q} = \int d\mathbf{v}\, \mathbf{v} f(\mathbf{v};\, \mathbf{x}), \tag{4}$$

and

$$3nT + n\mathbf{q}^2 = \int d\mathbf{v}\, \mathbf{v}^2 f(\mathbf{v};\, \mathbf{x}). \tag{5}$$

In this simplest of the hierarchy of statistical models (Fishman 1957, and Brau 1965), there is but a single free parameter, the collision frequency $\nu(\mathbf{x})$. This can be chosen so that the temperature dependence of the viscosity or heat conductivity of the pseudo-gas, as determined by applying the Chapman–Enskog procedure to the Krook kinetic equation, matches that of a particular real gas. We have assumed implicitly that the problem has been formulated in dimensionless form through a characteristic distance $\bar{R}$, number density $\bar{n}$, and temperature $\bar{T}$, from which we define a characteristic collision frequency $\bar{\nu}(\bar{n}, \bar{T})$ and a characteristic velocity $\bar{u} = \{k\bar{T}/m\}^{\frac{1}{2}}$, where k is Boltzmann's constant and m is the molecular mass. This choice of dependent and independent variables accounts for the form of F in equation (2) and the appearance in the kinetic equation of the Knudsen number λ defined by

$$\lambda = \bar{u}/\bar{R}\bar{\nu}. \tag{6}$$

The Krook kinetic equation embodies the conservation of mass, momentum, and energy and possesses an 'entropy'; however, its most important property for present purposes is the fact that the equation is non-linear only through the appearance of n, $\mathbf{q}$, and T in F. This means that f is completely determined by its low-order moments. In the case of the Maxwell–Boltzmann equation with molecular interactions described by the Boltzmann collision integral, f is specified by a finite number of its moments only in the case of thermodynamic equilibrium. It is essentially this property of the Krook equation which allows one to obtain numerically-exact solutions of a class of model problems. As we shall show in detail below, one can express f as a functional of F by integrating the kinetic equation along the characteristics of the convective differential operator. Substituting this formal expression into equations (3)–(5), we obtain a closed system of singular, non-linear integral equations for n, $\mathbf{q}$, and T. In the course of this calculation, boundary conditions must be assumed. The question of boundary conditions will be discussed at greater length in §5; suffice it to say for the present that we shall for simplicity consider a perfect accommodation boundary condition. Once the integral equations are solved for n, $\mathbf{q}$, and T, f and all of its higher moments are accessible, in particular, such quantities as the stress tensor $\mathbf{P}$ and heat flux vector $\mathbf{h}$ defined by

$$\mathbf{P} = \int d\mathbf{v}(\mathbf{v}-\mathbf{q})(\mathbf{v}-\mathbf{q})f(\mathbf{v};\,\mathbf{x}), \tag{7}$$

and

$$\mathbf{h} = \int d\mathbf{v}(\mathbf{v}-\mathbf{q})(\mathbf{v}-\mathbf{q})^2 f(\mathbf{v};\,\mathbf{x}). \tag{8}$$

A number of investigators have been, and are using the Krook kinetic equation as a physical tool in the study of particular physical phenomena (see, for example, de Leeuw 1965). While there is not yet enough evidence to suggest when this might be a reasonable approximation, there is evidence to the effect that the qualitative features of the phenomena are retained. In default of any adequate means of solving the Maxwell–Boltzmann equation, even qualitative results are of considerable interest. In the present work, however, we choose to adopt the attitude that the Krook kinetic equation is primarily a mathematical tool for investigating approximation procedures. In the case of steady problems in one space-dimension, two problems—the Couette flow with heat transfer and the structure of a plane shock wave—have been studied by a wide variety of approximation procedures. Restricting attention to the Krook kinetic equation, Willis (1960*a*, 1961, 1962, 1963) has considered the linearized Couette flow and heat-transfer problems, and Liepmann, Narasimha & Chahine (1926), Chahine (1963), and Chahine & Narasimha (1965) have considered the shock-structure problem. The integral-equation approach to these two problems was also considered by the author (Anderson 1963, 1965) in a collaborative effort with Prof. Max Krook and Dr H. K. Macomber to investigate the efficacy of a class of approximation procedures by comparing exact and approximate solutions of these model problems. The nature of the particular class of approximation procedures considered and comparisons of the results obtained are to be found in Anderson & Macomber (1965), and Macomber (1965). In brief, the comparison of exact and approximate solutions of these model problems suggests that, with sufficient insight into the problem at hand, relatively simple and reasonably accurate approximation procedures for solving non-linear problems in kinetic theory can be developed. These results offer some modicum of justification for the use of such procedures in problems whose exact solution is not feasible. There remain, however, certain conceptual and computational obstacles to the development of such procedures for non-steady and/or multidimensional problems. Consequently, extension of exact solutions of the Krook equation in these directions may aid in the development and evaluation of such procedures, and, incidentally, in the evaluation of the statistical model as a physical rather than a mathematical tool.

As indicated above, I have chosen to consider the generalization of the integral-equation formulation of the Krook kinetic equation to steady problems in two and three space-dimensions. A few words seem in order as to why non-steady problems in one space-dimension, which are in some respects easier and more interesting, were not chosen. Using variants of the Krook equation, Willis (1960*b*) has considered the formation of a plane shock wave by the impulsive motion of a piston, and Chu (1965) and Bienkowski (1965) have considered the formation of a plane shock wave from an initial density discontinuity. While the approaches used are rather different in each of the three cases, they have in common the problem of numerical stability endemic to initial-value problems. In such problems, the space and time discretizations used must reflect the intrinsic scale of the relevant phenomena as well as, and in relation to, the extrinsic scale of the problem. In the present context, the time discretization must be comparable with, or preferably smaller than, the smallest local mean collision time, rather than depending on a characteristic time based upon the motion of the shock wave. Such considerations essentially limit a straightforward numerical attack on the kinetic equation to a period of a relatively small number of mean collision times. It seems necessary to couple such a treatment of the initial layer on the microscopic time scale to a 'continuum' interior solution which varies on a macroscopic time scale. Such 'boundary layer' approaches are being investigated (see, for example, McCune, Sandri & Frieman 1963; McCune, Morse & Sandri 1963) but involve a substantial reconsideration of the whole framework in which the problem is to be studied. There is a strong analogy between the sequential solution of an initial-value problem and the conventional iterative solution of the corresponding boundary-value problem. The rate and mode of convergence of the initial-value problem to the steady state has its parallel in the rate and mode of convergence of the iterative process. The limits imposed by questions of stability in the initial-value problems are mirrored in problems of slow or non-existent convergence of the iterative process. On the other hand, the steady problem has the advantage that one is not interested in the intermediate results obtained as the calculation proceeds, but only in the final answer. Consequently, one can often accelerate the convergence of the iterative process by destroying the analogy with the initial-value problem. The generalization of steady problems to two or three space-dimensions seems then to offer the path of least resistance.

3. Cylindrical formation

Consider the Krook kinetic equation for a steady, two-dimensional problem. We regard $f(\mathbf{v};\, \mathbf{x})$ as a function of x_1 and x_2 but not of x_3, and, without loss of generality, define $q_3 \equiv 0$. Such a problem can be reduced to consideration of the

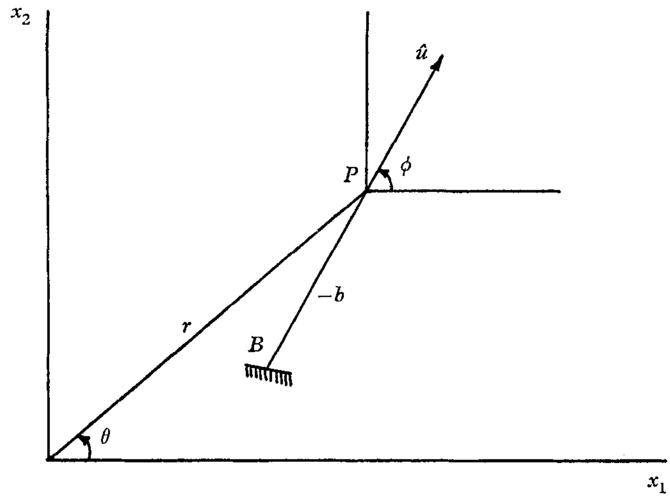

FIGURE 1. Cylindrical formulation.

$(x_3 = 0)$-plane, and we define $\mathbf{r}$ and $\mathbf{u}$ as the projections of $\mathbf{x}$ and $\mathbf{v}$ on this canonical plane. The Krook kinetic equation can then be written

$$\lambda \mathbf{u} . (\partial f / \partial \mathbf{r}) = \nu(F - f). \tag{9}$$

We shall obtain a formal solution of the equation by integrating along the characteristic line through $\mathbf{r}$ with direction $\mathbf{u}$.

Consider an arbitrary point $P{:}\mathbf{r}$ in configuration space, which corresponds to a point within the gas (see figure 1). For an arbitrary point $Q{:}\mathbf{u}$ in velocity space with $u > 0$, we define a characteristic line through P with direction $\hat{u} \equiv \mathbf{u}/u$. Define s as a displacement co-ordinate with origin $\mathbf{r}$ and direction $\hat{u}$. The restriction of the Krook kinetic equation (9) to the characteristic (P, Q) reads

$$\lambda u (d/ds) f(\mathbf{v};\, \mathbf{r} + s\hat{u}) = \nu(F - f). \tag{10}$$

Define a reduced co-ordinate t and collision frequency η by

$$t \equiv \eta s \equiv \int_0^s \nu(\mathbf{r}+s\hat{u})\,ds. \tag{11}$$

Then, equation (10) can be written

$$\lambda u(df/dt)+f = F, \tag{12}$$

or,

$$\frac{d}{dt}[e^{t/\lambda u} f] = \frac{1}{\lambda u} e^{t/\lambda u} F. \tag{13}$$

Let B be the first boundary point encountered on traversing the characteristic in the $-\hat{u}$-direction, through a distance $b(\mathbf{r}, \hat{u})$. Define

$$c(\mathbf{r}-b\hat{u}) = \eta(\mathbf{r}-b\hat{u})\,b,$$

a measure of the number of collisions a particle might suffer in traversing the characteristic. As discussed below, the distribution function f is assumed to be specified at B; hence we have

$$f(\mathbf{v};\,\mathbf{r}) = f(\mathbf{v};\,\mathbf{r}-b\hat{u})\,e^{-c/\lambda u}+\frac{1}{\lambda u}\int_0^c dt\,F(\mathbf{v};\,\mathbf{r}-s\hat{u})\,e^{t/\lambda u}, \tag{14}$$

where, for simplicity, we choose the perfect accommodation boundary condition

$$f(\mathbf{v};\,\mathbf{r}-b\hat{u}) = n_B/\{2\pi T_B\}^{\frac{3}{2}} \exp-\{(\mathbf{v}-\mathbf{q}_B)^2/2T_B\}. \tag{15}$$

It will prove convenient subsequently to express the vector quantities $\mathbf{x}, \mathbf{v}$ and $\mathbf{q}$ interchangeably in rectangular or cylindrical polar co-ordinates according to the equations

$$\left\{\begin{aligned} x_1 &= r\cos\theta,\\ x_2 &= r\sin\theta,\\ x_3 &= z, \end{aligned}\right\} \quad \left\{\begin{aligned} r &= \sqrt{(x_1^2+x_2^2)},\\ \theta &= \tan^{-1}(x_2/x_1),\\ z &= x_3; \end{aligned}\right\} \tag{16}$$

$$\left\{\begin{aligned} v_1 &= u\cos\phi,\\ v_2 &= u\sin\phi,\\ v_3 &= w, \end{aligned}\right\} \quad \left\{\begin{aligned} u &= \sqrt{(v_1^2+v_2^2)},\\ \phi &= \tan^{-1}(v_2/v_1),\\ w &= v_3; \end{aligned}\right\} \tag{17}$$

$$\left\{\begin{aligned} q_1 &= q\cos\psi,\\ q_2 &= q\sin\psi, \end{aligned}\right\} \quad \left\{\begin{aligned} q &= \sqrt{(q_1^2+q_2^2)},\\ \psi &= \tan^{-1}(q_2/q_1). \end{aligned}\right\} \tag{18}$$

Note that the problem is *not* being completely reformulated in cylindrical co-ordinates, since this would introduce geometric pseudo-force terms and complicate the definition of characteristics.

Since F is a functional of n, $\mathbf{q}$, and T—the low-order moments of f—(14) defines f only implicitly. However, substitution of the formal expressions for f, in terms of F and the boundary data, into the defining relations (3)–(5), yields a closed, coupled set of singular, non-linear integral equations for n, $\mathbf{q}$ and T. Once these integral equations are solved, f and all higher moments such as heat-flux vector and stress tensor components are available through (14), and the defining relations (7) and (8). As noted above, it is the fact that the distribution function is uniquely determined by specification of its low-order moments which allows numerically exact solutions of the Krook kinetic equation to be obtained.

To complete the reduction of the Krook kinetic equation for a steady, two-dimensional problem to integral equation form, we first summarize the relevant equations, and then proceed to reduce them. Substituting (2) and (15) in (14), we obtain

$$f(\mathbf{v};\,\mathbf{r}) = \frac{n_B(\mathbf{r}-b\hat{u})}{\{2\pi T_B(\mathbf{r}-b\hat{u})\}^{\frac{3}{2}}}\exp-\left\{\frac{[\mathbf{u}-\mathbf{q}_B(\mathbf{r}-b\hat{u})]^2+w^2}{2T_B(\mathbf{r}-b\hat{u})}+\frac{c(\mathbf{r}-b\hat{u})}{\lambda u}\right\}$$
$$+\frac{1}{\lambda u}\int_0^c dt\,\frac{n(\mathbf{r}-s\hat{u})}{\{2\pi T(\mathbf{r}-s\hat{u})\}^{\frac{3}{2}}}\exp-\left\{\frac{[\mathbf{u}-\mathbf{q}(\mathbf{r}-s\hat{u})]^2+w^2}{2T(\mathbf{r}-s\hat{u})}+\frac{t}{\lambda u}\right\}. \qquad (19)$$

Note that the parameters of the boundary term ($c, n_B, \mathbf{q}_B$, and T_B) and those of the collision term ($t, n, \mathbf{q}$, and T) are functions of $\mathbf{r}$ and $\hat{u}$ but not of u. For simplicity, the arguments of these quantities will be suppressed hereafter. Using the cylindrical polar co-ordinates in velocity space defined by (17), (3)–(5) can be written

$$n(\mathbf{r}) = \int_0^{2\pi} d\phi \int_0^\infty du\,u \int_{-\infty}^\infty dw f(\mathbf{v};\,\mathbf{r}), \qquad (20)$$

$$n\mathbf{q} = \int_0^{2\pi} d\phi\,\hat{u} \int_0^\infty du\,u^2 \int_{-\infty}^\infty dw f(\mathbf{v};\,\mathbf{r}), \qquad (21)$$

$$3nT+nq^2 = \int_0^{2\pi} d\phi \int_0^\infty du\,u^3 \int_{-\infty}^\infty dw f(\mathbf{v};\,\mathbf{r}) + \int_0^{2\pi} d\phi \int_0^\infty du \int_{-\infty}^\infty dw\,w^2 f(\mathbf{v};\,\mathbf{r}). \qquad (22)$$

Analogous expressions corresponding to (7) and (8) can be written down as required.

To continue the reduction, we shall carry out the $\int_{-\infty}^\infty dw$ operation explicitly, and reduce the $\int_0^\infty du$ operation to a kernel function which has been studied in connexion with steady, one-dimensional problems. The $\int_0^{2\pi} d\phi$ operation cannot be carried out explicitly due to the $\hat{u}$, or essentially ϕ, dependence of the boundary and collision terms of (19). The $\int_{-\infty}^\infty dw$ operations can be carried out by means of the identity

$$\int_{-\infty}^\infty dw\,e^{-w^2/2T}\,w^{2k}/\sqrt{(2\pi T)} = [1.3.5\ldots(2k-1)]\,T^k, \qquad (23)$$

for $k = 0, 1, \ldots$ and the usual convention that empty products are taken as unity. To treat the $\int_0^\infty du$ operation, we define

$$H_n(p,q) = \frac{1}{\sqrt{(2\pi)}}\int_0^\infty du\,u^{n-2}\exp-\{\tfrac{1}{2}(u-p)^2+q/u\}, \qquad (24)$$

for $-\infty < p < \infty$, $q > 0$, and $n = 1, 2, \ldots$. This function has been discussed in Anderson (1963) as well as Anderson & Macomber (1964), where properties, algorithms for the evaluation, and tables of the function are given. More generally, we define

$$K_n(\alpha,\beta,\gamma) = H_n(\alpha\cos\beta,\gamma)\exp-\tfrac{1}{2}\{\alpha\sin\beta\}^2, \qquad (25)$$

for $0 \leqslant \alpha < \infty$, $0 \leqslant \beta < 2\pi$, $0 < \gamma < \infty$, and $n = 1, 2, \ldots$. The factor $(\mathbf{u}-\mathbf{q})^2$ can be written in the form

$$(\mathbf{u}-\mathbf{q})^2 = u^2+q^2-2\mathbf{u}\,.\,\mathbf{q} \tag{26}$$

$$= [u-q\cos|\phi-\psi|]^2+[q\sin|\phi-\psi|]^2. \tag{27}$$

With these preliminaries, it is easily shown that

$$\int_0^\infty du \frac{u^l}{\sqrt{(2\pi T)}} \exp-\{(\mathbf{u}-\mathbf{q})^2/2T+t/\lambda u\} = T^{\frac{1}{2}l} K_{l+2}(q/\sqrt{T}, |\phi-\psi|, t/\lambda\sqrt{T}). \tag{28}$$

Finally, carrying out the

$$\int_{-\infty}^\infty dw \quad \text{and} \quad \int_0^\infty du$$

operations through (23) and (28), equations (20)–(22) can be reduced to the form

$$n(\mathbf{r}) = \frac{1}{\sqrt{(2\pi)}}\int_0^{2\pi} d\phi \left[n_B K_3(q_B/\sqrt{T_B}, |\phi-\psi_B|, c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt \frac{n}{\sqrt{T}} K_2(q/\sqrt{T}, |\phi-\psi|, t/\lambda\sqrt{T})\right], \tag{29}$$

$$n\mathbf{q} = \frac{1}{\sqrt{(2\pi)}}\int_0^{2\pi} d\phi\,\hat{u} \left[n_B \sqrt{T_B} K_4(q_B/\sqrt{T_B}, |\phi-\psi_B|, c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt\, n K_3(q/\sqrt{T}, |\phi-\psi|, t/\lambda\sqrt{T})\right], \tag{30}$$

and
$$3nT+nq^2 = \frac{1}{\sqrt{(2\pi)}}\int_0^{2\pi} d\phi \Big[n_B T_B\{K_5(q_B/\sqrt{T_B}, |\phi-\psi_B|, c/\lambda\sqrt{T_B}) + K_3(q_B/\sqrt{T_B}, |\phi-\psi_B|, c/\lambda\sqrt{T_B})\} + \frac{1}{\lambda}\int_0^c dt\, n\sqrt{T}\{K_4(q/\sqrt{T}, |\phi-\psi|, t/\lambda\sqrt{T}) + K_2(q/\sqrt{T}, |\phi-\psi|, t/\lambda\sqrt{T})\}\Big]. \tag{31}$$

Analogous expressions corresponding to (7) and (8) can be written down as required. Equations (29)–(31) constitute the cylindrical formulation of steady, two-dimensional problems with the Krook kinetic equation.

4. Spherical formulation

Consider the Krook kinetic equation for a steady, three-dimensional problem:

$$\lambda\mathbf{v}\,.\,(\partial f/\partial\mathbf{x}) = \nu[F-f]. \tag{32}$$

We shall obtain a formal solution of the equation by integrating along the characteristic line through $\mathbf{x}$ with direction $\mathbf{v}$. We shall exploit the similarities to the two-dimensional case considered in § 2 to abbreviate the discussion here.

Consider a point $P\colon\mathbf{x}$ in configuration space which corresponds to a point in the gas (see figure 2). For an arbitrary point $Q\colon\mathbf{v}$ in velocity space with $v > 0$, we define a characteristic line through P with direction $\hat{v} \equiv \mathbf{v}/v$. Define s as a displacement co-ordinate with origin $\mathbf{x}$ and direction $\hat{v}$. The restriction of the Krook kinetic equation (32) to the characteristic (P, Q) reads

$$\lambda v(d/ds) f(\mathbf{v};\, \mathbf{x}+s\hat{v}) = \nu(F-f). \tag{33}$$

Define a reduced co-ordinate t and collision frequency η by

$$t \equiv \eta s \equiv \int_0^s \nu(\mathbf{x}+s\hat{v})\,ds. \tag{34}$$

Then, equation (33) can be written

$$\lambda v(df/dt)+f = F, \tag{35}$$

or

$$\frac{d}{dt}(e^{t/\lambda v}f) = \frac{1}{\lambda v}e^{t/\lambda v}F. \tag{36}$$

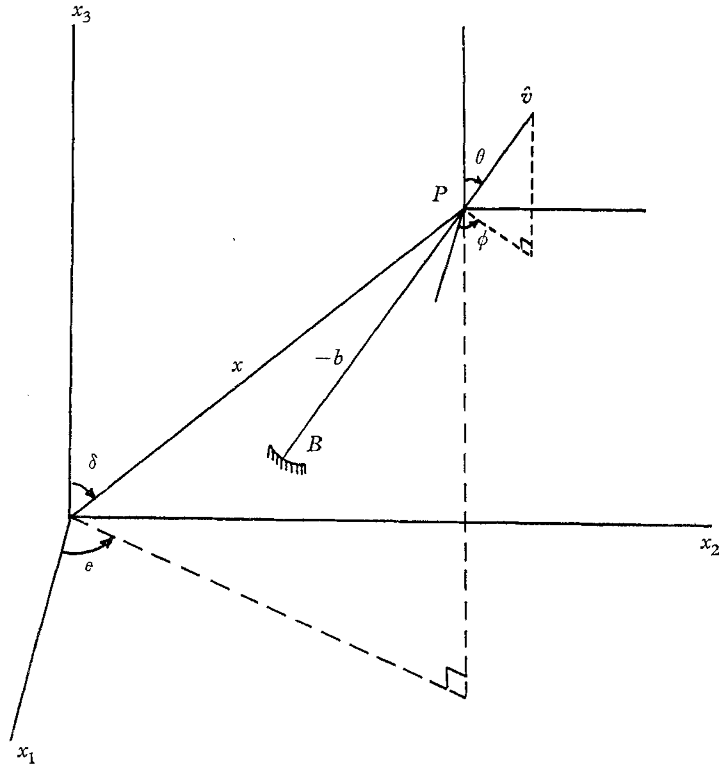

FIGURE 2. Spherical formulation.

Let B be the first boundary point encountered on traversing the characteristic in the $(-\hat{v})$-direction, through a distance $b(\mathbf{x},\hat{v})$. Define

$$c(\mathbf{x}-b\hat{v}) = \eta(\mathbf{x}-b\hat{v})\,b, \tag{37}$$

a measure of the number of collisions a particle might suffer in traversing the characteristic. As discussed below, the distribution function f is assumed to be specified at B; hence, we have

$$f(\mathbf{v};\,\mathbf{x}) = f(\mathbf{v};\,\mathbf{x}-b\hat{v})\,e^{-c/\lambda v}+\frac{1}{\lambda v}\int_0^c dt\,F(\mathbf{v};\,\mathbf{x}-s\hat{v})\,e^{t/\lambda v}, \tag{38}$$

where, for simplicity, we choose the perfect accommodation boundary condition

$$f(\mathbf{v};\,\mathbf{x}-b\hat{v}) = n_B/\{2\pi T_B\}^{\frac{3}{2}}\exp-\{(\mathbf{v}-\mathbf{q}_B)^2/2T_B\}. \tag{39}$$

It will prove convenient subsequently to express the vector quantities **x**, **v**, and **q** interchangeably in rectangular or spherical polar co-ordinates according to the equations

$$\begin{Bmatrix} x_1 = x\sin\delta\cos\epsilon, \\ x_2 = x\sin\delta\sin\epsilon, \\ x_3 = x\cos\delta, \end{Bmatrix} \quad \begin{Bmatrix} x = \surd(x_1^2+x_2^2+x_3^2), \\ \epsilon = \tan^{-1}(x_2/x_1), \\ \delta = \cos^{-1}(x_3/x), \end{Bmatrix} \tag{40}$$

$$\begin{Bmatrix} v_1 = v\sin\theta\cos\phi, \\ v_2 = v\sin\theta\sin\phi, \\ v_3 = v\cos\theta, \end{Bmatrix} \quad \begin{Bmatrix} v = \surd(v_1^2+v_2^2+v_3^2), \\ \phi = \tan^{-1}(v_2/v_1), \\ \theta = \cos^{-1}(v_3/v), \end{Bmatrix} \tag{41}$$

$$\begin{Bmatrix} q_1 = q\sin\zeta\cos\psi, \\ q_2 = q\sin\zeta\sin\psi, \\ q_3 = q\cos\zeta, \end{Bmatrix} \quad \begin{Bmatrix} q = \surd(q_1^2+q_2^2+q_3^2), \\ \psi = \tan^{-1}(q_2/q_1), \\ \zeta = \cos^{-1}(q_3/q). \end{Bmatrix} \tag{42}$$

Note that the problem is *not* being completely reformulated in spherical co-ordinates, since this would introduce geometric psuedo-force terms and complicate the definition of characteristics.

To complete the reduction of the Krook kinetic equation for a steady, two-dimensional problem to integral-equation form, we first summarize the relevant equations and then proceed to reduce them. Substituting (2) and (39) in (38), we obtain

$$f(\mathbf{v};\,\mathbf{x}) = \frac{n_B(\mathbf{x}-b\hat{v})}{\{2\pi T_B(\mathbf{x}-b\hat{v})\}^{\frac{3}{2}}}\exp-\left\{\frac{[\mathbf{v}-\mathbf{q}_B(\mathbf{x}-b\hat{v})]^2}{2T_B(\mathbf{x}-b\hat{v})}+\frac{c(\mathbf{x}-b\hat{v})}{\lambda v}\right\}$$
$$+\frac{1}{\lambda v}\int_0^c dt\,\frac{n(\mathbf{x}-s\hat{v})}{\{2\pi T(\mathbf{x}-s\hat{v})\}^{\frac{3}{2}}}\exp-\left\{\frac{[\mathbf{v}-\mathbf{q}(\mathbf{x}-s\hat{v})]^2}{2T(\mathbf{x}-s\hat{v})}+\frac{t}{\lambda v}\right\}. \tag{43}$$

Note that the parameters of the boundary term (c, n_B, $\mathbf{q}_B$, and T_B) and those of the collision term ($t, n, \mathbf{q}$, and T) are functions of $\mathbf{x}$ and $\hat{v}$ but not of v. For simplicity, the arguments of these quantities will be suppressed hereafter. Using the spherical polar co-ordinates in velocity spaces defined by (41), (3)–(5) can be written

$$n(\mathbf{x}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\sin\theta \int_0^{\infty} dv\, v^2 f(\mathbf{v};\,\mathbf{x}), \tag{44}$$

$$n\mathbf{q} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\sin\theta\,\hat{v} \int_0^{\infty} dv\, v^3 f(\mathbf{v};\,\mathbf{x}), \tag{45}$$

$$3nT+nq^2 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\sin\theta \int_0^{\infty} dv\, v^4 f(\mathbf{v};\,\mathbf{x}). \tag{46}$$

Analogous expressions corresponding to (7) and (8) can be written down as required.

To continue the reduction, we shall reduce the $\int_0^\infty dv$ operation to the K_n function defined above. The $\int_0^\pi d\theta$ and $\int_0^{2\pi} d\phi$ operations cannot be carried out explicitly, in general due to the $\hat{v}$, or essentially θ and ϕ, dependence of the integrand. The factor $(\mathbf{v}-\mathbf{q})^2$ can be written in the form

$$(\mathbf{v}-\mathbf{q})^2 = v^2+q^2-2\mathbf{v}\,.\,\mathbf{q} \tag{47}$$

$$= v^2+q^2-2vq(\hat{v}\,.\,\hat{q}). \tag{48}$$

Define an angle $\sigma(\hat{v}.\hat{q})$, such that $0 \leqslant \sigma < 2\pi$, by

$$\cos\sigma \equiv \hat{v}.\hat{q} = \sin\theta\sin\zeta\cos(\phi-\psi)+\cos\theta\cos\zeta. \tag{49}$$

With these preliminaries, it is easily shown that

$$\int_0^\infty dv \frac{v^l}{\sqrt{(2\pi T)}} \exp -\{(\mathbf{v}-\mathbf{q})^2/2T + t/\lambda v\} = T^{\frac{1}{2}l} K_{l+2}(q/\sqrt{T}, \sigma, t/\lambda\sqrt{T}). \tag{50}$$

Finally, carrying out the $\int_0^\infty dv$ operation through (50), equations (44)–(46) can be reduced to the form

$$n(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \left[n_B K_4(q_B/\sqrt{T_B}, \sigma_B, c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt \frac{n}{\sqrt{T}} K_3(q/\sqrt{T}, \sigma, t/\lambda\sqrt{T}) \right], \tag{51}$$

$$n\mathbf{q} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \hat{v} \left[n_B \sqrt{(T_B)} K_5(q_B/\sqrt{T_B}, \sigma_B, c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt\, n K_4(q/\sqrt{T}, \sigma, t/\lambda\sqrt{T}) \right], \tag{52}$$

and

$$3nT + nq^2 = \frac{1}{2\pi}\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \left[n_B T_B K_6(q_B/\sqrt{T_B}, \sigma_B, c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt\, n\sqrt{T}\, K_5(q/\sqrt{T}, \sigma, t/\lambda\sqrt{T}) \right]. \tag{53}$$

Analogous expressions corresponding to (7) and (8) can be written down as required. Equations (51)–(53) constitute the cylindrical formulation of steady, three-dimensional problems with the Krook kinetic equation.

5. Boundary conditions

A particular steady-flow problem is specified by defining the interaction of gas molecules with material boundaries and the geometric configuration under consideration. In this section, we shall consider the question of boundary conditions, postponing the formulation of a particular problem until the next section.

Conceptually, we restrict consideration to interior problems, those in which all characteristics terminate on a bounding surface at some finite distance b. We choose to regard exterior problems, such as the aligned flow over a flat strip to be considered below, as a limiting situation in which the conceptual outer boundary recedes to infinity while all other problem parameters are held fixed. For finite λ, in the limit as $b \to \infty$, the boundary term in equation (19) or (43) tends to zero exponentially, and the free-stream conditions 'at infinity' enter the problem only implicitly through assumptions on the asymptotic behaviour of the solution. The convention, that $b \to \infty$ only after λ is specified, is established, in part, to remove the ambiguity of the so called 'free molecular limit', $\lambda \to \infty$ in such exterior problems.

If the characteristic terminates on a boundary at some finite distance b, we must specify the distribution of particles emitted from the surface in the direction defined by the characteristic. In many respects, the simplest and most natural boundary condition is that of perfect accommodation: molecules are assumed to be emitted from the boundary with a Maxwellian velocity distribution characterized by the local temperature and velocity of the boundary. Symbolically, the perfect accommodation boundary condition states

$$f(\mathbf{v};\, \mathbf{x}-b\hat{v}) = F_B(\mathbf{v}) = n_B/\{2\pi T_B\}^{\frac{3}{2}} \exp -\{(\mathbf{v}-\mathbf{q}_B)^2/2T_B\}, \tag{54}$$

where $\mathbf{q}_B$ and T_B are the local boundary velocity and temperature, respectively. n_B is a parameter which must be determined as part of the solution of the problem so as to assure zero net mass flux normal to the boundary, that is, $\mathbf{q}\,.\hat{N}_B = 0$, where $\mathbf{q}$ is the local fluid velocity adjacent to B and $\hat{N}_B$ is the unit normal vector to the bounding surface at B. Usually, $\mathbf{q}_B$ and T_B will be specified *ab initio*, but they could be determined as part of the problem if equations defining the dynamic and thermodynamic characteristics of the boundary were adjoined.

As will appear shortly, perfect accommodation is the easiest boundary condition to work with; moreover, this would seem to be the natural boundary condition to associate with the Krook kinetic equation. The statistical model of molecular interactions, which constitutes the right side of equation (1), can be interpreted as a statement to the effect that those particles in a given phase space element $d\mathbf{x}d\mathbf{v}$ which have suffered a collision during the time interval dt may be characterized, on the average, by a local Maxwellian velocity distribution. On the other hand, a perfect accommodation boundary condition could be interpreted as a similar statement concerning molecule–boundary collisions. For these reasons of simplicity and naturalness, the perfect accommodation boundary condition was used in the plane Couette flow with heat-transfer problem considered previously, and in the generalizations to be considered in the present work.

It is well known, however, that a perfect accommodation boundary condition is but a rough, though reasonable, approximation to physical reality; the same might be said of the Krook kinetic equation. A somewhat more sophisticated, though still over-simplified, boundary condition is Maxwell's partial accommodation boundary condition. In this case, it is assumed that the flux emitted from the boundary is a superposition of two species of molecules: first, a class of fully thermalized or accommodated molecules, and second, a class of specularily reflected molecules. Symbolically, such a boundary condition would read

$$f(\mathbf{v};\, \mathbf{x}-b\hat{v}) = aF_B+(1-a)f_R, \tag{55}$$

where a is an accommodation coefficient, F_B is the boundary Maxwellian defined above, and f_R is the distribution of specularily reflected molecules incident at B with velocity $\mathbf{v}-2\mathbf{v}\,.\hat{N}_B$. Clearly a formal expression for f_R can be obtained by specializing (19) or (43) above along the characteristic through B with direction $\mathbf{v}-2\mathbf{v}\cdot\hat{N}_B$ and terminating at some other boundary point B', which must then be treated in analogous fashion. The final equation, corresponding to (19) or (43), obtained with this partial accommodation boundary condition is considerably complicated by the fact that the significant history of a particle extends beyond the time of its last encounter with the boundary. The equation will, in general, consist of an infinite sequence of integrals over a broken line sequence of characteristics. Conceptually, a more general boundary condition involving the integrated effect of all characteristics emanating from B could be formulated, but the resulting equations would be far too cumbersome to work with in practice. Indeed, even partial accommodation is amenable to treatment only in geometrically simple configurations. For simplicity, we restrict consideration in the present work to the case of total accommodation ($a = 1$).

To investigate the effect of accommodation coefficients in the typical range $0{\cdot}75 \leqslant a \leqslant 1$ on the solutions of the Krook kinetic equation, it is proposed shortly to repeat certain of the plane Couette flow with heat transfer calculations using the more general boundary conditions. Because of the simplicity of the plane parallel geometry, the complexity of the governing equations is not unduly increased. Although they are intrinsically more complex by virtue of their geometry, exterior problems, such as the aligned flow over a flat strip to be discussed below, can relatively easily be generalized from total to partial accommodation boundary conditions. For a flow around a convex body, the reflected characteristic extends to infinity and no complicated multiple-reflexion pattern is involved.

6. Formulation of a particular problem

In this section, we shall formulate a specific example of a steady problem with the Krook kinetic equation. Choosing a total accommodation boundary condition, such a formulation reduces to the definition of the geometric configuration, the associated geometric factors b and $\hat{N}_B$, and the boundary data $\mathbf{q}_B$ and T_B. Certain computational aspects of the resulting equations will be discussed briefly; however, the description of detailed numerical procedures and solutions, as well as the choice of specific models for the collision frequency ν, will be deferred to subsequent papers of this series in which a sequence of specific problems will be discussed.

A steady problem of great historical and experimental interest, and one of the simplest problems geometrically, is the aligned flow over a finite strip. This problem is typical of a class of exterior, two-dimensional cylinder flows described by equations (29)–(31). A close cousin of the finite-strip problem which is somewhat similar but more complicated, is the axisymmetric flow around a sphere described by equations (51)–(53). For the sake of brevity, only the finite-strip problem will be formulated here, but the sphere problem is readily accessible. These two problems involve a wide range of flow phenomena: slip, boundary-layer growth, wakes, edge and curvature effects, and even shock structure. As a result, such problems represent probably the *pièce de résistance* rather than the *hors d'oeurve* in any investigation of the integral formulation of the Krook kinetic equation. Nevertheless, some of the computational questions involved can be surveyed at this stage.

Consider the aligned flow over a flat strip in a frame of reference fixed with respect to the strip, as shown schematically in figure 3. We choose the characteristic quantities $\bar{R}$, $\bar{n}$, $\bar{T}$, and $\bar{\nu}$ with respect to which the problem is made dimensionless, so that the strip is of unit width and so that $\mathbf{r}.\hat{x}_1 \to -\infty$ implies that $n \to T \to \nu \to 1$ and $\mathbf{q} \to Q\hat{x}_1$. By construction, we have $\mathbf{q}_B \equiv 0$. The simplest version of the problem is that in which T_B takes on some given constant value across the strip. However, one could consider a more complicated situation in which T_B assumes a constant value across the strip which is to be determined as part of the problem, or the even more complicated situation in which T_B is allowed to vary across the strip in a manner to be determined from an additional equation describing the heat flow through the strip. Due to the symmetry of the problem we restrict attention to the half plane $x_2 > 0$, whence $\hat{N}_B = \hat{x}_2$. Thus, n_B is to be determined as a function of position on the strip so that $q_2 = 0$ for $0 \leqslant x_1 \leqslant 1$.

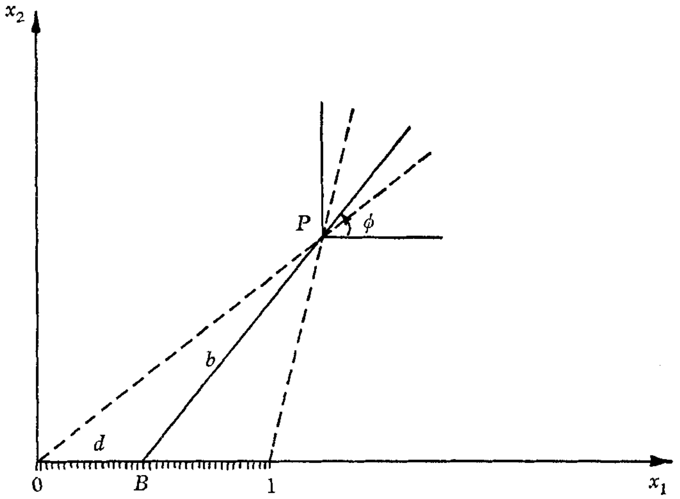

FIGURE 3. Finite-strip problem.

Consider the characteristic line through $P: (x_1, x_2)$ with direction ϕ, whose intersection with the $x_2 = 0$ axis is the point $(d, 0)$, where

$$d = x_2 - x_1 \cot\phi. \tag{56}$$

If $d(1-d) > 0$, the characteristic terminates on the strip with finite b given by

$$b = \sqrt{\{x_2^2 + (x_1 - d)^2\}}. \tag{57}$$

If $d(1-d) < 0$, the characteristic extends to infinity (recall the convention established above for the meaning attributed to $b = \infty$).

Defining

$$G_n(\gamma) \equiv H_n(0,\gamma) \equiv K_n(0,\beta,\gamma), \tag{58}$$

we can summarize the equations for the finite-strip problem as

$$n(\mathbf{r}) = \frac{1}{\sqrt{(2\pi)}}\int_0^{2\pi} d\phi \left[n_B G_3(c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt \frac{n}{\sqrt{T}} K_2(q/\sqrt{T}, |\phi - \psi|, t/\lambda\sqrt{T})\right], \tag{59}$$

$$n\mathbf{q} = \frac{1}{\sqrt{(2\pi)}}\int_0^{2\pi} d\phi\, \hat{u}\left[n_B T_B G_4(c/\lambda\sqrt{T_B}) + \frac{1}{\lambda}\int_0^c dt\, n K_3(q/\sqrt{T}, |\phi - \psi|, t/\lambda\sqrt{T})\right], \tag{60}$$

and $$3nT+nq^2 = \frac{1}{\sqrt{(2\pi)}}\int_0^{2\pi} d\phi\left[n_BT_B\{G_5(c/\lambda\sqrt{T_B})+G_3(c/\lambda\sqrt{T_B})\}\right.$$
$$+\frac{1}{\lambda}\int_0^c dt\, n\sqrt{T}\,\{K_4(q\sqrt{T},|\phi-\psi|,t/\lambda\sqrt{T})$$
$$\left.+K_2(q/\sqrt{T},|\phi-\psi|,t/\lambda\sqrt{T})\}\right]. \tag{61}$$

The numerical solution of these equations is a formidable undertaking. The wide variety of flow phenomena referred to above must be reflected in the structure of the numerical procedures employed, raising many knotty computational questions. At this stage, we shall simply comment briefly on several aspects of the problem which are accessible without delving too deeply into the numerical analysis.

It is of great practical, if not intellectual, import that the kernel function $K_n(\alpha,\beta,\gamma)$, appearing in the equations derived above, can be regarded as a known quantity. A substantial fraction of the effort expended on the one-dimensional Couette flow with heat transfer and shock-structure problems was devoted to the study of the properties of the G_n and H_n functions which arise in these problems. Consequently, we are now in a position to comment on the nature of the kernel function arising in the general two- and three-dimensional problems, as formulated here. We note first that the lowest order, n, of the kernel functions entering the equations is equal to the dimensionality of the problem: 1 for the plane Couette flow with heat transfer and shock-structure problems, 2 for the finite-strip problem, and 3 for the sphere problem. Since we have

$$K_n(\alpha,\beta,\gamma) = O(\gamma^{n-1}\ln\gamma), \quad \text{as} \quad \gamma\to 0, \tag{62}$$

the most singular kernel involved is logarithmically unbounded for $n = 1$ but bounded and progressively less singular for $n = 2$ and 3. The nature of the singularities of the kernel functions involved in the equations is seen to be an artifact of the geometry of the problem under consideration.

In the limit $\gamma\to\infty$, $K_n(\alpha,\beta,\gamma)\to 0$ exponentially rapidly, but at a rate which is strongly dependent upon α and β. The component function $H_n(p,\gamma)$ is a very asymmetric function of p: for fixed γ, H_n varies as p^{n-2} for large positive p but as $e^{-\frac{1}{2}p^2}$ for large negative p. In addition, the rate of decay of H_n with γ is a decreasing function of p and of n. If we imagine the topography of $K_n(\alpha,\beta,\gamma)$, for fixed α, over a plane with polar co-ordinates γ and β, we see that the bulk of the kernel is concentrated near the ray $\beta = 0$ and extends in γ over a distance which is an increasing function of α and n. This 'focusing' effect is most relevant in the supersonic régime, where α is large, but is present in some measure in all problems involving fluid motion. While K_n is a strictly monotonic decreasing function of γ, the effective kernel function appearing in the equations will not in general be monotonic due to the implicit dependence of α, β and γ on the variable of integration. Clearly, the integrals to be performed in (59)–(61) are considerably more complicated than appears on the surface.

A second salient feature of the class of steady problems with the Krook kinetic equation is the so-called 'curse of dimensionality': the fact that the labour involved in solving a problem tends to increase exponentially rather than algebraically with the dimensionality of the problem. The computing time required to solve a system of non-linear integral equations such as those derived above can be roughly characterized by a number M which is the product of three factors:

M_1, the number of quadrature sample points used to approximate the integral operators;
M_2, the number of interpolation sample points used to approximate the dependent variables, and
M_3, the number of iterations required to obtain an answer to the discrete analog of the analytic problem within a prescribed residual.

For purposes of comparison, let a typical one-dimensional problem like the Couette flow with heat transfer or shock-structure problems be characterized by the triplet (m_1, m_2, m_3). The corresponding triplets for the finite strip and sphere problems would be roughly (m_1^2, m_2^2, m_3^2) and (m_1^3, m_2^2, m_3^2) respectively. Typically, one would have $100 \gtrsim m_1 \gtrsim m_2 \gtrsim m_3 \gtrsim 10$. Consideration of an exterior rather than an interior problem will, in general, increase M_1 and M_2, and indirectly M_3. M_1 would be substantially increased if a partial accommodation boundary condition were used, and both M_2 and M_3 might be expected to increase with the complexity of the sub-structures of the flow, for example, in the presence of shock waves. One can make the crude estimate $M_3 \approx M_2$; while this may be pessimistic for large λ where convergence tends to be quite rapid, it may be realistic or even optimistic for $\lambda \lesssim 1$, depending on the type of iterative procedure employed. These are but qualitative estimates of M; nevertheless, it is clear that there is a drastic increase in computational complexity in going from one-dimensional to multi-dimensional problems.

Of similar import is another feature of the transition from one to several space-dimensions. In steady, one-dimensional problems, one can obtain solutions which are universal in the sense that the free parameter of the statistical model, the collision frequency ν, can be suppressed. This can be done by defining a reduced independent variable t proportional to the integral of ν along the characteristic direction; since there is only one such direction, the problem can be solved wholly in terms of t, with the inverse transformation from t to s being relegated to a subsidiary *a posteriori* calculation. The use of the reduced variable t as the independent variable reduces the computation time required by roughly an order of magnitude. Unfortunately, in the two- and three-dimensional equations formulated above, t is defined along individual characteristics and cannot be used as the fundamental independent variable. The transformation between s and t is implicitly contained in the equations in a fashion which must be made explicit at the time of solution. One can carry out the $\int_0^c dt$ operations as they stand or in the form $\int_0^b ds\,\nu$. In the first case, the s–t transformation is involved in the arguments of the dependent variables appearing in the integrand, and in the second case, in the argument of the kernel function. The first choice distributes the quadrature sample points optimally with respect to the natural local scale of the problem; however, the second choice is favoured by the greater facility with which the transformation from t to s rather than the inverse transformation from s to t can be carried out. No universal solutions, in the sense defined above, are possible; however, one should clearly solve the analogous reduced problem $\nu \equiv \eta \equiv 1$ first, in which case $t \equiv s$. The essential features of the flow are determined by the geometry, the degree of rarefaction (as reflected in λ), and the conservation laws, and not by the details of the collision processes. Only when there are large gradients in the macroscopic dependent variables will the profiles be greatly altered by the dependence of ν on n and T. After this fundamental set of solutions is obtained, the effects of variable ν can be inserted as perturbations. This should mitigate the drastic order of magnitude increase in computation time occasioned by the appearance of the s–t transformation in the equations.

One final point should be made, even though it is concerned more with numerical analysis than kinetic theory. Generally speaking, one solves a non-linear continuous problem like the integral equations considered here by replacing the analytic problem by a discrete analogue and solving the resulting finite system of non-linear algebraic or transcendental equations. Only after analysing how accurately the discrete problem has been solved can one investigate the relationship of the answer obtained to the solution of the original problem. In linear problems, a well-posed analytic problem usually results in a well-posed discrete problem; in non-linear problems, it is not true, in general that a well-posed analytic problem leads to a well-posed discrete problem. The 'solution' of the discrete problem may fail to exist or to be unique, and the 'answer' obtained may simply satisfy the equations 'best', in some sense. Such an occurrence severely complicates any attempt at error analysis; indeed, one is often faced with the phenomenon, familiar in the study of asymptotic series, of a lower bound on the accuracy which a given numerical procedure can provide. There is evidence to suggest that such considerations are operative in some of the procedures proposed for the solution of the integral equations of interest here.

The problem of the aligned flow over a finite strip is probably accessible with present day computers; however, it would be naïve to begin an investigation of the integral formulation of the steady Krook kinetic equation for multi-dimensional problems with such a potentially complex problem. Rather, one must first consider a sequence of simpler problems which isolate certain properties of the full equations in more tractable form, hoping that the experience gained will enable one to tackle more complicated problems.

A prime candidate for such a sequence of simpler problems is the class of cylindrical analogues of the Couette flow with heat-transfer problem considered previously. The primary reason for this choice is the simplification induced by the symmetry and interior character of this geometry. However, rotating fluids are of some interest in their own right, and the cylindrical geometry is more amenable to experimental simulation than the plane parallel geometry. We shall outline below a set of problems of this class, indicating the features of the general equations isolated in each. Calculations are substantially completed for the first problem, and will be reported shortly.

The first problem to be considered is the pure heat-transfer problem. The complexity of the cylindrical geometry, as reflected in equations (29)–(31), is somewhat mitigated by the azimuthal symmetry, which reduces M_2 and, hopefully, M_3, but not M_1. In this problem we can isolate the geometric features of the problem and the complexity of the required quadratures, while eschewing the added complication of the K_n kernel for the relative simplicity of the G_n kernel. The corresponding heat-transfer problem for concentric spheres is described by essentially the same equations as the cylindrical case, and the two heat-transfer problems are conveniently treated in parallel.

The second problem to be considered is the degenerate Couette flow problem in which the inner cylinder is absent. The geometry is simplified in order to investigate two new features of the problem: the effect of local variations in the degree of rarefaction and of the flow velocity dependence of the K_n kernel functions. In the interior of a rapidly rotating cylinder, one can span a substantial portion of the transition flow régime, through the density variation induced by centrifugal forces. The azimuthal symmetry of the flow field allows one to investigate conveniently the computational implications of the 'focusing effect' inherent in the K_n kernel functions.

Finally, the combined Couette flow with heat-transfer problem can be studied, including various special cases of limiting geometric configurations and boundary conditions. With this backlog of experience it may be feasible to tackle at least the subsonic finite-strip problem.

7. Concluding remarks

The stage has been set for the solution of a sequence of steady problems in two and three space-dimensions with the integral-equation formulation of the Krook kinetic equation. Some background and motivation for the study, the derivation of the general equations, and a discussion of some of the features of these equations have been given. Specialization to particular problems and the details of the numerical procedures employed will be forthcoming shortly.

This work was supported in part by the National Science Foundation under Grants GP-414 and GK-65, and by the Division of Engineering and Applied Physics, Harvard University.

REFERENCES

ANDERSON, D. G. 1963 Numerical experiments in kinetic theory. Ph.D. Thesis, Harvard University, Cambridge, Massachusetts.

ANDERSON, D. G. 1965 *Tech. Rep.* no. 15, Engineering Sciences Laboratory, Harvard University, Cambridge, Massachusetts (to be published in *J. Fluid Mech.*).

ANDERSON, D. G. & MACOMBER, H. K. 1964 *Tech. Rep.* no. 10, Engineering Sciences Laboratory, Harvard University, Cambridge, Massacusetts (to be published in *J. Math. & Phys.*).

ANDERSON, D. G. & MACOMBER, H. K. 1965 In *Proc. 4th Int. Symp. on Rarefied Gas Dynamics* (ed. J. H. de Leeuw). New York: Academic Press.

BIENKOWSKI, G. 1965 In *Proc. 4th Int. Symp. on Rarefied Gas Dynamics* (ed. J. H. de Leeuw). New York: Academic Press.

BRAU, C. A. 1965 Models for collision processes in polyatomic gases, Ph.D. Thesis, Harvard University, Cambridge, Massachusetts.

CHAHINE, M.T. 1963 In *Proc. 3rd Int. Symp. on Rarefied Gas Dynamics* (ed. J. A. Laurmann). New York: Academic Press.

CHAHINE, M. T. & NARASIMHA, R. 1965 In *Proc. 4th Int. Symp. on Rarefied Gas Dynamics* (ed. J. H. de Leeuw). New York: Academic Press.

CHU, C. K. 1965 *Phys. Fluids*, **8**, 12.

FISHMAN, F. G. 1957 A kinetic approach to neutral and ionized gas dynamics, Ph.D. Thesis, Harvard University, Cambridge, Massachusetts.

KROOK, M. 1955 *Phys. Rev.* **99**, 1896.

KROOK, M. 1959 *J. Fluid Mech.* **6**, 523.

LEEUW, DE J. H. (ed.) 1965 *Proc. 4th Int. Symp. on Rarefied Gas Dynamics.* New York: Academic Press.

LIEPMANN, H. W., NARASIMHA, R. & CHAHINE, M. T. 1962 *Phys. Fluids*, **5**, 1313.

MACOMBER, H. K. 1965 Moment methods for non-linear problems in kinetic theory, Ph.D. Thesis, Harvard University, Cambridge Massachusetts. (Also available as *Tech. Rep.* no. 13, Engineering Sciences Laboratory, Harvard University, Cambridge, Massachusetts.)

MCCUNE, J. E., MORSE, T. F. & SANDRI, G. 1963 *Proc. 3rd Int. Symp. on Rarefied Gas Dynamics*, (ed. J. A. Laurmann). New York: Academic Press.

MCCUNE, J. E., SANDRI, G. & FRIEMAN, E. A. 1963 *Proc. 3rd Int. Symp. on Rarefied Gas Dynamics* (ed. J. A. Laurmann). New York: Academic Press.

WILLIS, D. R. 1960*a* *Proc. 1st Int. Symp. on Rarefied Gas Dynamics* (ed. F. M. Devienne). New York: Pergamon Press.

WILLIS, D. R. 1960*b* *Tech. Note* no. 4, Division of Gas Dynamics, Royal Institute of Technology, Stockholm, Sweden.

WILLIS, D. R. 1961 *Proc. 2nd Int. Symp. on Rarefied Gas Dynamics.* (ed. L. Talbot). New York: Academic Press.

WILLIS, D. R. 1962 *Phys. Fluids*, **5**, 127.

WILLIS, D. R. 1963 *Proc. 3rd Int. Symp. on Rarefied Gas Dynamics* (ed. J. A. Laurmann. New York: Academic Press.